Sydney Technical High School

Student Name ______________________________

Teacher's Name: ______________________________

Extension 1 Mathematics

TRIAL HSC

August 2023

General Instructions

- Reading time – 10 minutes
- Working time – 120 minutes
- Write using black pen
- NESA approved calculators may be used
- A NESA reference sheet is provided with this paper
- In questions 11-14, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II – 60 marks

- Attempt questions 11-14
- Allow about 1 hour and 45 minutes for this section

SECTION I

10 marks

Attempt Questions 1-10.

Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1-10.

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1 The number of different arrangements of the letters of the word SPUDDLE which begins with the letter P is:

A. $\dfrac{7!}{2!}$

B. $\dfrac{6!}{2!}$

C. $\dfrac{6!}{5!\,2!}$

D. $\dfrac{7!}{5!\,2!}$

2 What is the remainder when $P(x) = x^4 - 3x^3 + 6x^2 - 2$ is divided by $(x - 1)$?

A. 2

B. -4

C. 8

D. 12

3 Which of the following vectors is parallel to the vector $\overrightarrow{OQ} = 3\underset{\sim}{i} - \underset{\sim}{j}$?

A. $\overrightarrow{OA} = \underset{\sim}{i} - 3\underset{\sim}{j}$

B. $\overrightarrow{OB} = 12\underset{\sim}{i} - 6\underset{\sim}{j}$

C. $\overrightarrow{OC} = 3\underset{\sim}{i} + \underset{\sim}{j}$

D. $\overrightarrow{OD} = -12\underset{\sim}{i} + 4\underset{\sim}{j}$

4 Given that $\tan\alpha = \frac{3}{4}$ and $\tan\beta = \frac{5}{12}$, what is the value of $\cos(\alpha + \beta)$?

A. $\frac{63}{65}$

B. $\frac{33}{65}$

C. $\frac{56}{65}$

D. $\frac{16}{65}$

5 A spherical balloon is filled with air at a rate of 108 π cm^3/ min. At what rate is the radius of the balloon increasing when the radius is 3 cm ?

A. 3.6 cm / min

B. 3 cm / min

C. 1 cm / min

D. 4 cm / min

6 Given that $f(x) = \ln(x - 2)$ what are the domain and range of $f^{-1}(x)$?

A. $x > 2$, all real y

B. $x < 2$, all real y

C. all real x , $y > 2$

D. all real x , $y < 2$

7 The parametric equation of a function is given by $x = \sin t$ and $y = \sin 2t$.

What is its cartesian equation

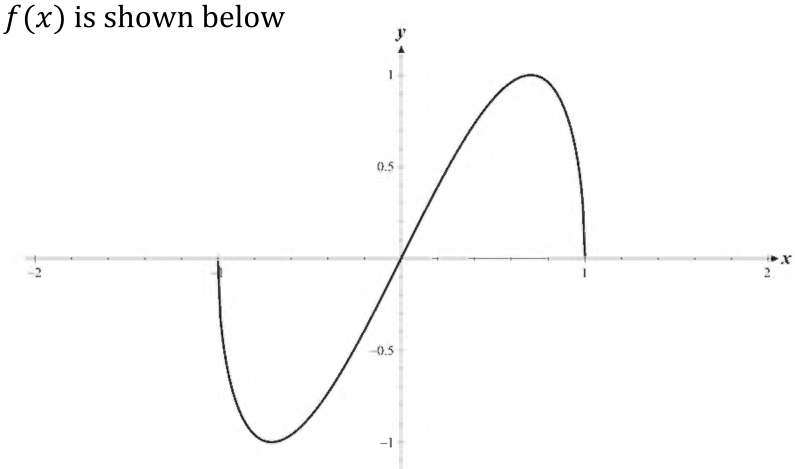
A. $y = 2x\sqrt{1 - x^2}$ $\{0 \le x \le 1\}$

B. $y = 2x^2 - 1$ $\{0 \le x \le 1\}$

C. $y = 1 - 2x^2$ $\{0 \le x \le 1\}$

D. $y = \sqrt{\dfrac{x+1}{2}}$ $\{0 \le x \le 1\}$

8 The graph of $y = f(x)$ is shown below



$f(x)$ could be

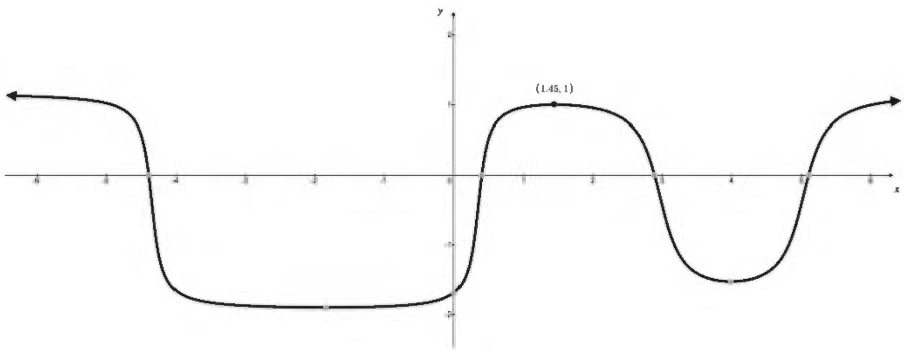
A. $\sin(-2 \arcsin x)$

B. $\arcsin(-2 \sin x)$

C. $\sin(2 \arcsin x)$

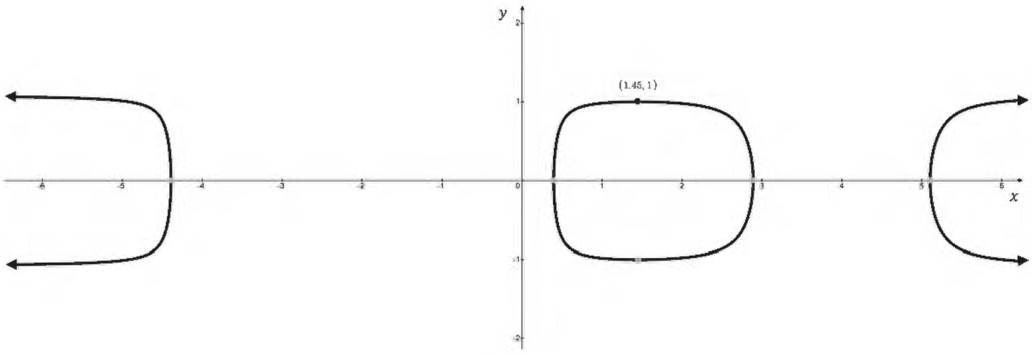
D. $\arcsin(2 \sin x)$

9 The graph of $y = f(x)$ is shown below

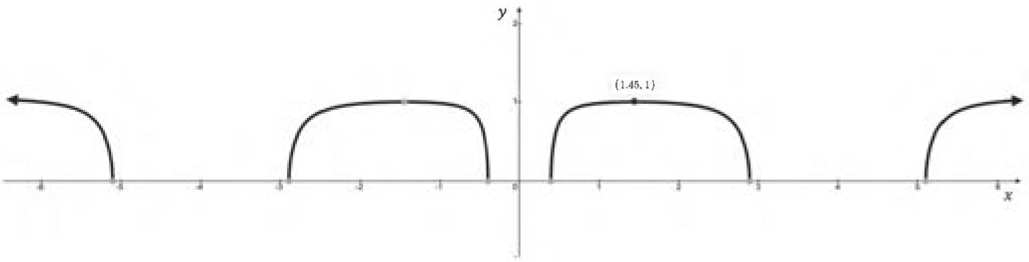


Which of the following could be the graph of $y^2 = f(|x|)$

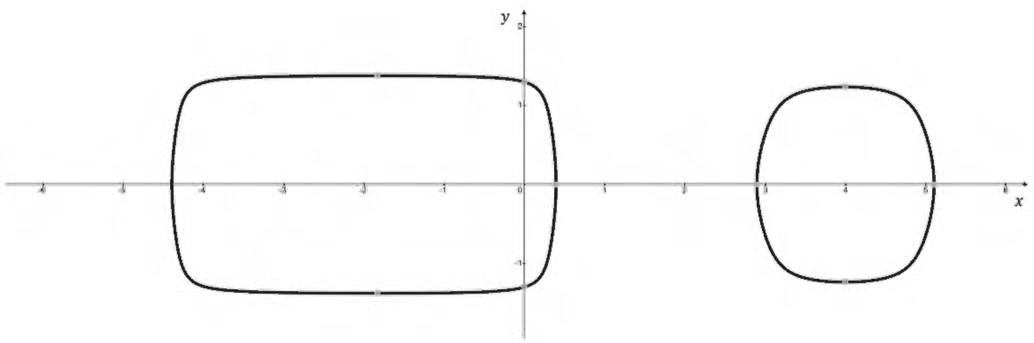
A.



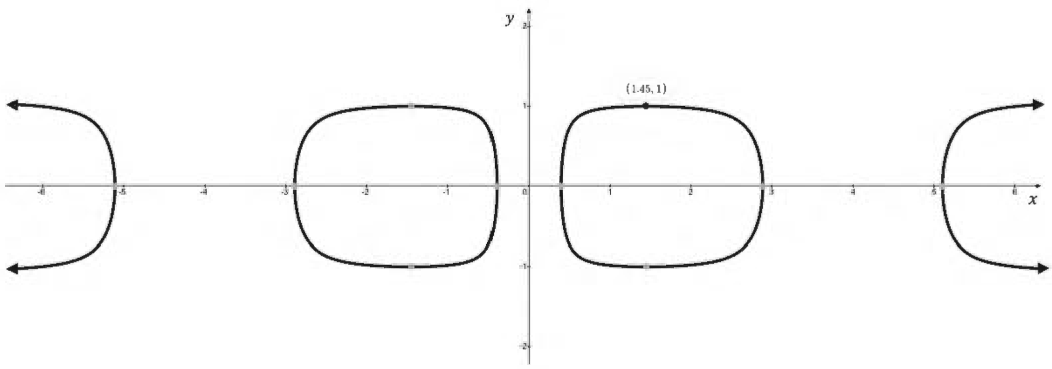
B.



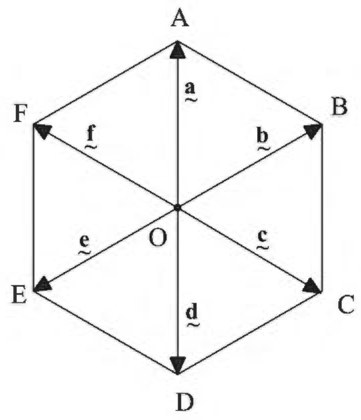
C.



D.



10 $ABCDEF$ is a regular hexagon made from 6 equilateral triangles with side-lengths of 2 units as shown below. $\overrightarrow{OA} = \underset{\sim}{\boldsymbol{a}}$, $\overrightarrow{OB} = \underset{\sim}{\boldsymbol{b}}$, $\overrightarrow{OC} = \underset{\sim}{\boldsymbol{c}}$, $\overrightarrow{OD} = \underset{\sim}{\boldsymbol{d}}$, $\overrightarrow{OE} = \underset{\sim}{\boldsymbol{e}}$, and $\overrightarrow{OF} = \underset{\sim}{\boldsymbol{f}}$



What is the value of $\text{proj}_{\underset{\sim}{\boldsymbol{c}}}\, \underset{\sim}{\boldsymbol{a}} \cdot \text{proj}_{\underset{\sim}{\boldsymbol{f}}}\, \underset{\sim}{\boldsymbol{e}}$ (the dot product of the vectors formed by projecting $\underset{\sim}{\boldsymbol{a}}$ and $\underset{\sim}{\boldsymbol{e}}$ onto $\underset{\sim}{\boldsymbol{c}}$ and $\underset{\sim}{\boldsymbol{f}}$ respectively) ?

A. -2

B. -1

C. 1

D. 2

End of Section I

SECTION II

60 marks

Attempt Questions 11-14.

Allow about 1 hour and 45 minutes for this section.

Answer each question on a new page in the answer booklet.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a **NEW** page

a) For the vectors $\underset{\sim}{u} = \underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{v} = 2\underset{\sim}{i} - \underset{\sim}{j}$, evaluate the following

i) $\underset{\sim}{u} + 3\underset{\sim}{v}$ **1**

ii) $\underset{\sim}{u} \cdot \underset{\sim}{v}$ **1**

b) Show that $\sin(75°) = \dfrac{\sqrt{6}+\sqrt{2}}{4}$ **2**

c) Differentiate $8\tan^{-1}\left(\dfrac{x}{2}\right)$ **2**

d) Solve $\dfrac{5x}{x+1} \leq 4$ **3**

e) Find the exact value of $\displaystyle\int_0^3 x\sqrt{x^2+16}\ dx$ by using the substitution $u = x^2 + 16$ **3**

f) For what value(s) of a are the vectors $\begin{pmatrix} 2a \\ -6 \end{pmatrix}$ and $\begin{pmatrix} a-3 \\ 6 \end{pmatrix}$ perpendicular **3**

End of Question 11

Question 12 (16 marks) Start a **NEW** Page

a) Given that $X \sim Bin(n, p), \mu = 12$ and $\sigma^2 = 4$

i) Show that $p = \frac{2}{3}$ **2**

ii) Hence find n **1**

iii) By considering a six-sided die or otherwise, describe a possible real-world scenario that could be represented by this binomial distribution. **1**

iv) Find $P(X = 4)$ **2**

b) The 4 distinct roots of the polynomial $x^4 + 4x^3 - x^2 - 16x - 12 = 0$ are $\alpha, \beta, \gamma, and\ \delta$ **2**

Find the value of $\frac{1}{\alpha\beta\gamma} + \frac{1}{\alpha\beta\delta} + \frac{1}{\alpha\gamma\delta} + \frac{1}{\beta\gamma\delta}$

c) Use mathematical induction to prove that **3**

$$4 \times 1! + 9 \times 2! + \cdots + (n+1)^2 \times n! = (n+2)! - 2$$

for all integers $n \geq 1$

d) The multiple-choice section of the Mathematics Extension 1 HSC exam consists of 10 questions, each with 4 options. Assuming students do not leave any question unanswered, what is the minimum number of candidates that must sit the test to ensure that there are at least 3 identical sets of answers? **2**

e) The student representative council at a particular Sydney high school consists of 42 students, 5 from each of the Years 7-10, 10 from Year 11, and 12 from Year 12.

i) A subcommittee of 18 students is needed to research the positive affect of banning mobile phones from schools. It will consist of 5 students each from Years 11 & 12 (for a total of 10), and 2 students each from the other Years 7-10 (for a total of 8). **1**

In how many ways can this subcommittee be formed ?

ii) In how many ways can the subcommittee sit around a circular table, if the five Year 12 students are to all sit together ? **2**

End of Question 12

Question 13 (15 marks) Start a **NEW** Page

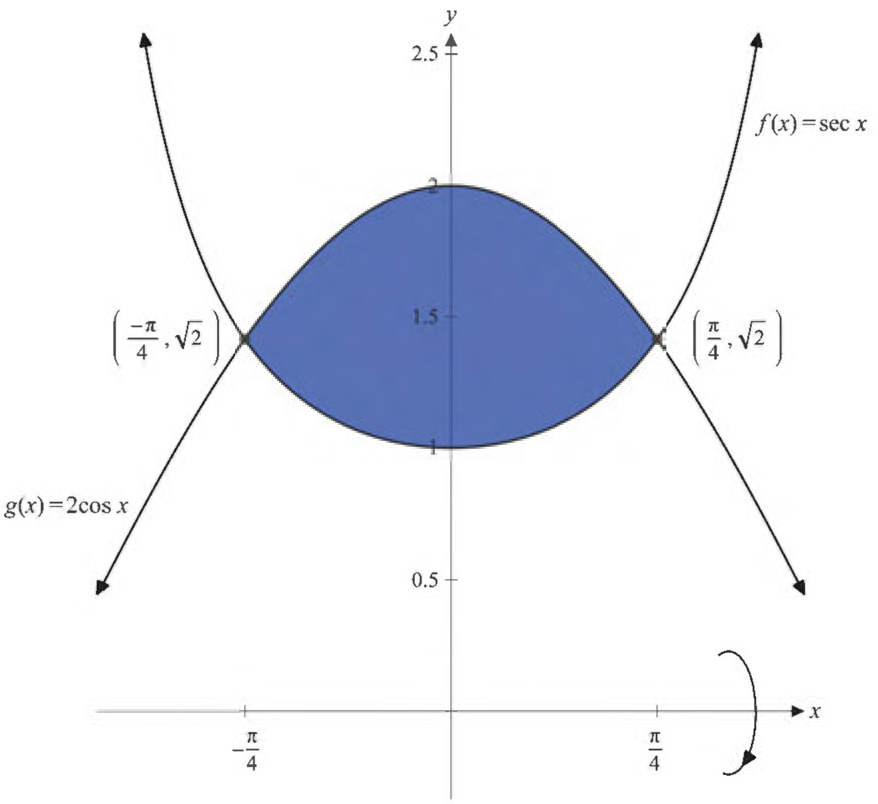
a) Solve the differential equation $\frac{dy}{dx} = 3(y+2)\sqrt{x}$, given that $y(0) = 7$ **3**

Give your answer in the form $y = f(x)$

b) Express $y = 2\cos x + 2\sqrt{3}\sin x$ in the form $R\cos(x - \alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$ **2**

c) Part of the graph of $f(x) = \sec x$ and $g(x) = 2\cos x$, which intersect when **3**

$x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$ is shown below



Find the exact volume of the solid formed when the shaded area bounded

by $f(x)$ and $g(x)$ is rotated about the $x - axis$.

d) Using research and years of empirical data, Mr Harroothunian has determined that only $\frac{1}{3}$ of the espresso coffees made in Melbourne's coffee shops meet his exacting standards. During his holidays, Mr Harroothunian travelled to Melbourne and randomly sampled 50 coffees.

Letting $\hat{p}$ represent the sample proportion of coffees that pleased Mr Harroothunian by meeting his standards, and assuming that the sample is normally distributed:

i) Find the mean and standard deviation of the sample **2**

ii) By using the table below or otherwise, find the probability that Mr Harroothunian was pleased with at least 40% of the coffees that he sampled **2**

Table of values $P(Z \leq z)$ for the normal distribution $N(0,1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
	....	....	....	....	....	....	....	....	....	....
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
....	....	....	....	....	....	....	....	....	....	....

e) Given that $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$ (do not show this) **3**

and by selecting two appropriate values of x, show that:

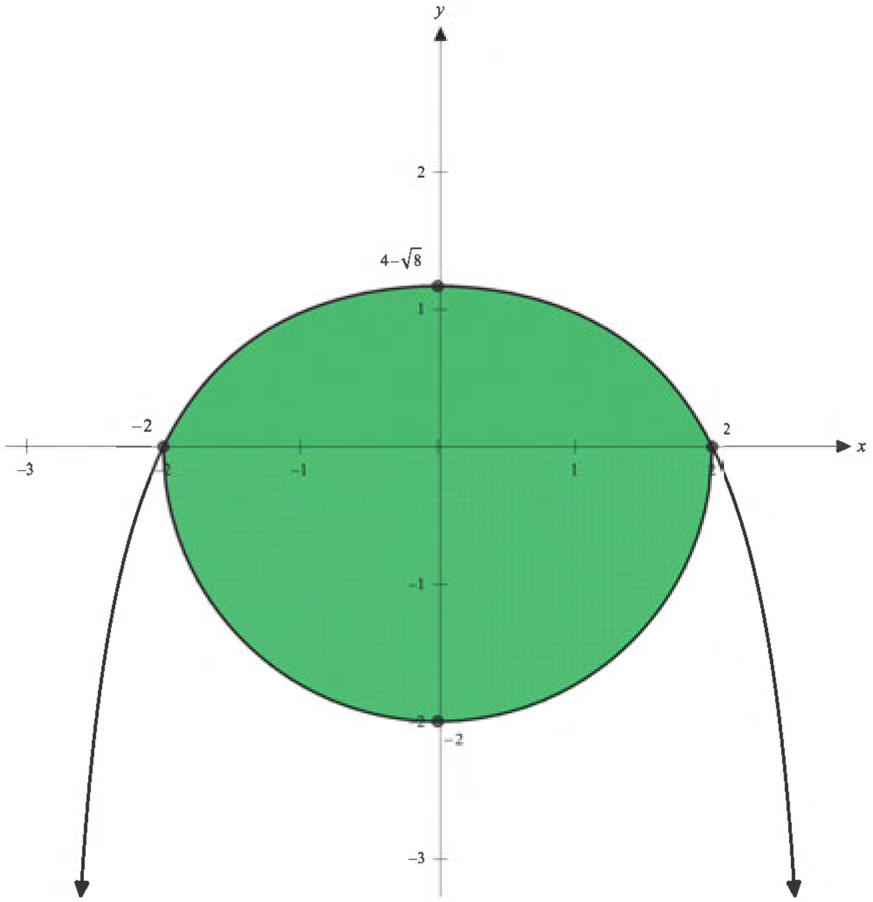
$$2^{n-1} = \binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{n-1} \quad \text{when } n \text{ is odd.}$$

End of Question 13

Question 14 (14 marks) Start a **NEW** Page

a) Use the product rule to show that $y = x\int f(x)dx$ is a solution to the differential equation $x\dfrac{dy}{dx} = y + x^2 f(x)$ **1**

b) The region enclosed by $y = 4 - \dfrac{8}{\sqrt{8 - x^2}}$ and the semi-circle $y = -\sqrt{4 - x^2}$ is shaded in the diagram **3**



Find the exact value of the area of the shaded region.

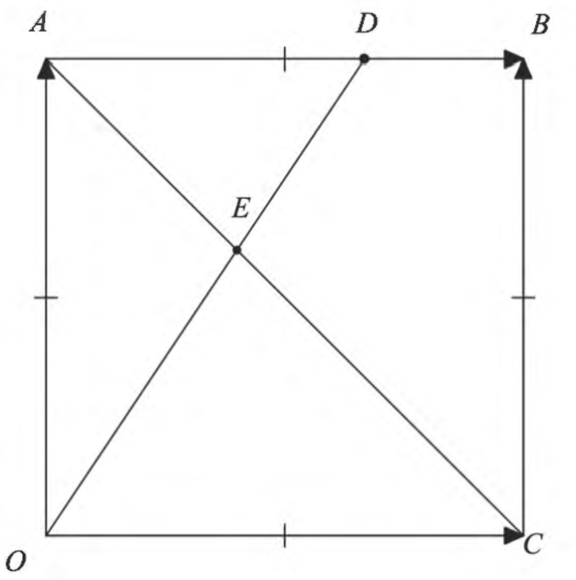
Question 14 (continued)

c) Let $f(x) = \dfrac{x}{(x+e)^2}$

i) Use the substitution $u = x + e$ to show that $\displaystyle\int_0^e f(x)\,dx = \int_e^{2e} \frac{u-e}{u^2}\,du$ **2**

ii) Hence find the exact value of $\displaystyle\int_0^e f(x)\,dx$ in its simplest form. **2**

d) $OABC$ is a square with $\overrightarrow{OA} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$



i) Find $\overrightarrow{OB}$ **1**

ii) Given that D is a point on AB such that $\overrightarrow{AD} = \frac{2}{3}\overrightarrow{AB}$, find $\overrightarrow{OD}$ **2**

iii) Given that OD intersects AC at E and that $\overrightarrow{OE} = (1-\lambda)\overrightarrow{OA} + \lambda\overrightarrow{OC}$, find λ **3**

End of paper

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2023 EX1 TRIAL

MC	1	2	3	4	5	6	7	8	9	10
	B	A	D	B	B	C	A	C	D	C

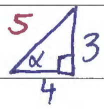
① P _ _ _ _ _ _ $\frac{1\times 6!}{2!}$ B

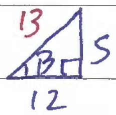
② $P(1) = 1^4 - 3\times1^3 + 6\times1^2 - 2$

$= 1 - 3 + 6 - 2 = \underline{\underline{2}}$ A

③ $\vec{OP} = 3\underset{\sim}{i} - \underset{\sim}{j}$ $\quad \vec{OD} = -12\underset{\sim}{i} + 4\underset{\sim}{j}$

$= -4(3\underset{\sim}{i} - \underset{\sim}{j})$ D

④ 



$\cos(\alpha+B) = \cos\alpha\cos B - \sin\alpha\sin B$

$= \frac{4}{5}\times\frac{12}{13} - \frac{3}{5}\times\frac{5}{13}$

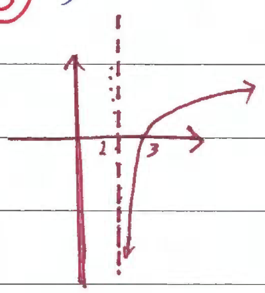
$= \frac{33}{65}$ B

⑤ $\frac{dV}{dt} = 108\pi \text{ cm}^3/\text{min}$ $\quad \frac{dr}{dt} = ?$ $\quad V = \frac{4}{3}\pi r^3$ $\quad \therefore \frac{dV}{dr} = 4\pi r^2$

$\frac{dV}{dt} = \frac{dV}{dr}\times\frac{dr}{dt}$ $\quad \therefore \frac{dr}{dt} = \frac{\frac{dV}{dt}}{\frac{dV}{dr}} = \frac{108\pi}{4\pi(3)^2}$ $\quad r=3$

$= 3\text{cm/min}$ B

⑥ $f(x) = \ln(x-2)$



$f(x)$
$D: x>2$
$R: \text{all } y$

$f^{-1}(x)$
$D: \text{all } x$
$R: y>2$

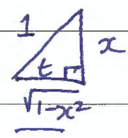
C

⑦ $y = \sin 2t$ $\quad x = \sin t$

$= 2\sin t\cos t$

$= 2x\cos t$

$= 2x\sqrt{1-x^2}$ A



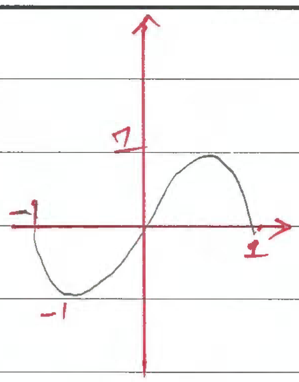
$\sin^2 x + \cos^2 x = 1$

$\cos^2 x = 1 - \sin^2 x$.

$\cos x = \sqrt{1-\sin^2 x}$

$= \underline{\sqrt{1-x^2}}$

(8)



→ note that we eliminate B and D as they would result in periodic functions.

→ also as $x \to 0$ $\arcsin(\text{small } x)$ is positive.

$\therefore$ must be (C)

→ you can also sub in points (trial and error)

(9) $y^2 = f(|x|)$ must be symmetrical around the y-axis (because of the $|x|$) and the y-axis (the y^2)

$\therefore$ (D) the only one that is

(10) $\text{Proj}_{\underset{\sim}{c}} \underset{\sim}{a} = |\underset{\sim}{a}||\hat{\underset{\sim}{c}}| \cos\angle AOC \times \hat{\underset{\sim}{c}}$

$= 2 \times 1 \times \cos 120^\circ \times \hat{\underset{\sim}{c}}$

$= 2 \times -\frac{1}{2}\, \hat{\underset{\sim}{c}}$

$= -\hat{\underset{\sim}{c}}$ which is the same as $\hat{\underset{\sim}{f}}$ as $\angle AOC = 180^\circ$.

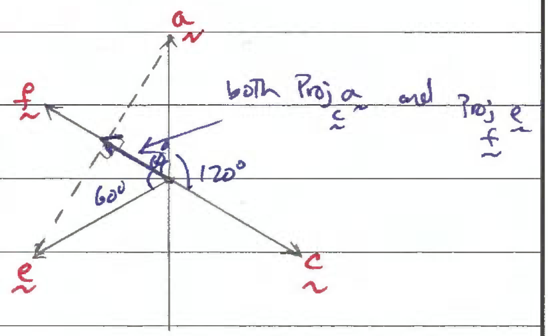
$= \hat{\underset{\sim}{f}}$

$\text{Proj}_{\underset{\sim}{f}} \underset{\sim}{e} = |\underset{\sim}{e}||\hat{\underset{\sim}{f}}| \cos\angle EOF \times \hat{\underset{\sim}{f}}$

$= 2 \times 1 \times \cos 60^\circ \times \hat{\underset{\sim}{f}}$

$= 2 \times \frac{1}{2} \times \hat{\underset{\sim}{f}}$

$= \hat{\underset{\sim}{f}}$



So $\text{Proj}_{\underset{\sim}{c}} \underset{\sim}{a} \cdot \text{Proj}_{\underset{\sim}{f}} \underset{\sim}{e}$ is just $= \hat{\underset{\sim}{f}} \cdot \hat{\underset{\sim}{f}}$ (dot product of 2 unit vectors)

$= 1$

$\therefore$ (C)

11

(a) $\underset{\sim}{u} = \underset{\sim}{i} + \underset{\sim}{j}$ $\quad \underset{\sim}{v} = 2\underset{\sim}{i} - \underset{\sim}{j}$

(i) $\underset{\sim}{u} + 3\underset{\sim}{v} = (\underset{\sim}{i} + \underset{\sim}{j}) + 3(2\underset{\sim}{i} - \underset{\sim}{j})$

$= 7\underset{\sim}{i} - 2\underset{\sim}{j}$

(ii) $\underset{\sim}{u} \cdot \underset{\sim}{v} = 2 - 1$

$= 1$

(b)

$\sin 75^\circ = \sin(30^\circ + 45^\circ)$

$= \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ$

$= \frac{1}{2} \times \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2}$

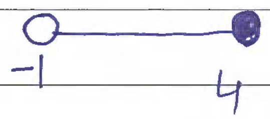
$= \frac{\sqrt{2} + \sqrt{6}}{4}$ ✓ as required

(c) $\frac{d}{dx}\left(8\tan^{-1}\left(\frac{x}{2}\right)\right) = 8 \times \frac{1}{2} \times \frac{1}{1 + \left(\frac{x}{2}\right)^2}$

$= \frac{4}{1 + \frac{x^2}{4}}$ ✓ or $\times \frac{4}{4}$ $= \frac{4 \times 4}{\left(1 + \frac{x^2}{4}\right) \times 4} = \frac{16}{4 + x^2}$

(d)

$\frac{5x}{x+1} \leq 4$ $\qquad x \neq 1 \longrightarrow$



now solve.

$\frac{5x}{x+1} = 4 \Rightarrow 5x = 4x + 4 \quad \therefore x = 4 \leftarrow$ critical point.

test a point $x = 0$, $\frac{5 \times 0}{1} \leq 4$ ✓ $\therefore$ soln between $-1, 4$.

i.e $-1 < x \leq 4$

(11e) $\int_{x=0}^{3} x\sqrt{x^2+16}$

$u = x^2+16$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$x=0 \rightarrow u=16$

$x=3 \quad u=25$

$= \frac{1}{2}\int_0^3 \sqrt{x^2+16}\cdot 2x\,dx$

$= \frac{1}{2}\int_{u=16}^{u=25} \sqrt{u}\cdot du = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_{16}^{25}$

$= \left[\frac{(\sqrt{u})^3}{3}\right]_{16}^{25} = \frac{5^3-4^3}{3} = \frac{61}{3}$

(f) $\begin{pmatrix} 2a \\ -6 \end{pmatrix}\cdot\begin{pmatrix} a-3 \\ 6 \end{pmatrix} = 0$ (For vectors to be $\perp$)

$2a^2-6a-36=0$

$a^2-3a-18=0$

$(a-6)(a+3)=0 \quad \therefore a=6$ or $a=-3$

Question 12

(a) $X \sim Bin(n,p)$ $\mu=12$, $\sigma^2=4$.

(i) $\mu = np$ and $\sigma^2 = npq$ $(q=1-p)$

$np=12$ — ① $npq=4$ ②

$\frac{②}{①}$ $\frac{npq}{np} = \frac{4}{12} = \frac{1}{3}$ $\therefore q=1/3$ and $P=2/3$ ✓ as required.

(ii) $np=12$ $n=\frac{12}{p} = 12\times 3/2 = 18$

(iii) $X\sim Bin(18, 2/3)$ Rolling a six-sided die 18 times and finding the probability of Rolling a # > 2

(iv) $P(X=4) = {}^{18}C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^{14} = 0.000126\ldots$

POS

Question 12 ...

(b) $x^4 + 4x^3 - x^2 - 16x - 12 = 0$

$\alpha+\beta+\gamma+\delta = -\frac{b}{a} = -4$

$\alpha\beta\gamma\delta = \frac{e}{a} = -12$

$\frac{1}{\alpha\beta\gamma} + \frac{1}{\alpha\beta\delta} + \frac{1}{\alpha\gamma\delta} + \frac{1}{\beta\gamma\delta}$

$= \frac{\delta+\gamma+\beta+\alpha}{\alpha\beta\gamma\delta} = \frac{-4}{-12} = \frac{1}{3}$

(c) RTP $4\times1! + 9\times2! + \cdots + (n+1)^2\times n! = (n+2)! - 2$

For $n=1$ $\quad$ LHS $= 4\times1!$ $= 4$ $\quad$ RHS $= (1+2)! - 2 = 3! - 2 = 4$

$\therefore$ true for $n=1$

Assume true for $n=k$, $k\in\mathbb{Z}^+$

i.e. $4\times1! + 9\times2! + \cdots + (k+1)^2\times k! = (k+2)! - 2$

RTP $4\times1! + 9\times2! + \cdots + (k+1)^2\times k! + (k+2)^2\times(k+1)! = (k+3)! - 2$

LHS $= 4\times1! + 9\times2! + \cdots + (k+1)^2\times k! + (k+2)^2\times(k+1)!$

$= (k+2)! - 2 + (k+2)^2(k+1)!$ (from assumption)

$= (k+2)! + (k+2)(k+2)(k+1)! - 2$

$= (k+2)! + (k+2)(k+2)! - 2$

$= (1+(k+2))(k+2)! - 2$

$= (k+3)(k+2)! - 2$

$= (k+3)! - 2$

$=$ RHS

$\therefore$ if true for $n=k$, then true for $n=k+1$

but true for $n=1$, $\therefore$ true for $n=2,3,\ldots$

$\therefore$ true $\forall n\in\mathbb{Z}^+$

Q12 continued.

(d) 10 Q's, 4 options   This is a pigeonhole or 'worst case scenario' qn.

need at least 3 identical set of answers.

worst case 1st unique set $= 4^{10} = 1048576$

2nd unique set $= 4^{10} = 1048576$.

+ 1 more (minimum # of candidates)

$\therefore = 4^{10} + 4^{10} + 1 = 2097153$

(e) 42 students Y7 : 5 Y8 : 5 Y9 : 5 Y10 : 5 Y11 : 10 Y12 : 12

(i) $\overset{Y7}{{}^5C_2} \times \overset{Y8}{{}^5C_2} \times \overset{Y9}{{}^5C_2} \times \overset{Y10}{{}^5C_2} \times \overset{Y11}{{}^{10}C_5} \times \overset{Y12}{{}^{12}C_5}$

$= ({}^5C_2)^4 \times {}^{10}C_5 \times {}^{12}C_5 = 199584000$ ✓

(ii) Fix the Y12 students together and seat them first. (so now we have 14 items in a circle.)

$= 1 \times 5! \times 13!$ = Pretty large ... ✓

Question 13

(a) $\frac{dy}{dx} = 3(y+2)\sqrt{x}$ $y(0) = 7$.

$\int \frac{dy}{y+2} = \int 3\sqrt{x}\,dx = \int 3x^{1/2}\,dx$.

$\ln|y+2| = \frac{3}{3/2}x^{3/2} + C = 2x^{3/2} + C$

$|y+2| = e^{(2x^{3/2}+C)}$

$y+2 = \pm e^{C} \cdot e^{2x^{3/2}}$

 P07

Question 13. (a) ... alternatively.

$y = \pm e^{c} e^{2x^{3/2}} - 2$

$= Ae^{2x^{3/2}} - 2 \quad (A = \pm e^{c})$

$7 = Ae^{0} - 2 \quad @(0,7)$

$\therefore A = 9$

$\therefore y = 9e^{2x^{3/2}} - 2$ ✓ or $y = e^{(2x^{3/2} + \ln 9)} - 2$ ✓ or $y = e^{2(x^{3/2} + \ln 3)} - 2$

(b) $y = 2\cos x + 2\sqrt{3}\sin x$

also

$y = R\cos(x - \alpha) = R(\cos x \cos\alpha + \sin x \sin\alpha)$

Resolving coefficients for $\cos x$ and $\sin x$.

$R\cos\alpha = 2$ ①

$R\sin\alpha = 2\sqrt{3}$ ② ∴ ② ÷ ①

$\tan\alpha = \frac{2\sqrt{3}}{2} = \sqrt{3} \quad \therefore \alpha = \pi/3 \quad [0, \pi/2]$

sub into ①.

$R\cos\pi/3 = 2 \quad \frac{R}{2} = 2 \quad \therefore R = 4$

$\therefore 2\cos x + 2\sqrt{3}\sin x = 4\cos(x - \pi/3)$ i.e. $y = 4\cos(x - \pi/3)$

(c) $V = \pi\left(\int_{-\pi/4}^{\pi/4} (g(x))^2\,dx - \int_{-\pi/4}^{\pi/4} (f(x))^2\,dx\right)$ as $g(x) \geq f(x)$ over the interval $[-\pi/4, \pi/4]$

$= \pi\int_{-\pi/4}^{\pi/4} (2\cos x)^2\,dx - \pi\int_{-\pi/4}^{\pi/4} \sec^2 x\,dx$

$= 2\pi\int_{0}^{\pi/4} 4\cos^2 x\,dx - 2\pi\int_{0}^{\pi/4} \sec^2 x\,dx.$

Question B(c)...

$$V = 4\pi \int_0^{\pi/4} (1+\cos 2x)\,dx - 2\pi \int_0^{\pi/4} \sec^2 x\,dx$$

$$= 4\pi \left[x + \frac{\sin 2x}{2}\right]_0^{\pi/4} - 2\pi \left[\tan x\right]_0^{\pi/4}$$

$$= 4\pi\left(\left(\frac{\pi}{4} + \frac{\sin(\pi/2)}{2}\right) - \left(0 + \frac{\sin 0}{2}\right)\right) - 2\pi\left(\tan\frac{\pi}{4} - \tan 0\right)$$

$$= \pi^2 + 2\pi - 2\pi$$

$$V = \pi^2 \ u^3$$

(d) $p = \frac{1}{3}$ $q = \frac{2}{3}$ $n = 50$

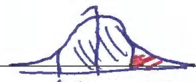
$E(\hat{p}) = \frac{np}{n}$ standard deviation $= \sqrt{Var(\hat{p})}$

$= p$ $= \sqrt{\frac{pq}{n}}$

$= \frac{1}{3}$ $= \sqrt{\frac{1/3 \times 2/3}{50}} = \sqrt{\frac{1}{225}}$

$= \frac{1}{15}$

(ii) Z-score $= \frac{0.4 - 1/3}{1/15} = 15 \times \frac{1}{15} = 1$



OR $P(Z \geq 1) = P(1 - P(Z \leq 1))$

$= 1 - 0.8413$

$= 15.87\%$

$P(\hat{p} \geq 0.4) = P(Z \geq 1) = 16\%$ (empirical rule)

(e) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$

Pick $x = 1$ and $x = -1$ as we need cancellation.

$(1+1)^n = \binom{n}{0} + \binom{n}{1}\cdot 1 + \binom{n}{2}(1)^2 + \cdots \binom{n}{n-1}(1)^{n-1} + \binom{n}{n}(1)^n$ — ①

$(1-1)^n = \binom{n}{0} + \binom{n}{1}\cdot(-1) + \binom{n}{2}(-1)^2 + \cdots + \binom{n}{n-1}(-1)^{n-1} + \binom{n}{n}(-1)^n$

$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + \binom{n}{n-1} - \binom{n}{n}$ — ② (as n is odd).

① + ②

$2^n + 0 = \binom{n}{0} + \binom{n}{0} + \binom{n}{1} - \binom{n}{1} + \binom{n}{2} + \binom{n}{2} - \cdots + \binom{n}{n-1} + \binom{n}{n-1} + \binom{n}{n} - \binom{n}{n}$

$2^n = 2\left(\binom{n}{0} + \binom{n}{2} + \cdots \binom{n}{n-1}\right)$

$2^{n-1} = \binom{n}{0} + \binom{n}{2} + \cdots \binom{n}{n-1}$ as required.

Question 14

(a)

$$y = x\int f(x)\,dx$$

$u = x$ $\quad v = \int f(x)\,dx$

$du = 1$ $\quad dv = f(x)$

$$\frac{dy}{dx} = v\,du + u\,dv =$$

$$\frac{dy}{dx} = \int f(x)dx \cdot 1 + x \cdot f(x)$$

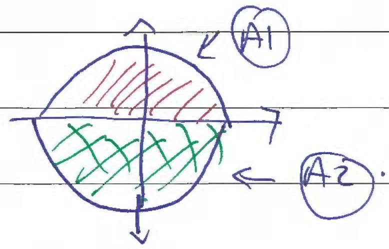
$$= \int f(x)dx + xf(x).$$

$$x\frac{dy}{dx} = x\left(\int f(x)dx + xf(x)\right) = x\int f(x)\,dx + x^2 f(x)$$

($x\int f(x)\,dx = y$)

$$x\frac{dy}{dx} = y + x^2 f(x) \quad \therefore \quad y = x\int f(x)\,dx \text{ is a solution to the DE}$$

(b)



$$A = A_1 + A_2$$

$$= \int_{-2}^{2}\left(4 - \frac{8}{\sqrt{8-x^2}}\right)dx + \text{area semi-circle of radius } 2$$

$$= 2\int_0^2\left(4 - \frac{8}{\sqrt{8-x^2}}\right)dx + \tfrac{1}{2}\pi \times 2^2$$

$$= 2\int_0^2\left(4 - 8 \times \frac{1}{\sqrt{8-x^2}}\right)dx + 2\pi$$

$$= 2 \times \left[4x - 8\arcsin\left(\frac{x}{2\sqrt{2}}\right)\right]_0^2 + 2\pi$$

$$= 2 \times \left[\left(4\times 2 - 8\arcsin\left(\frac{2}{2\sqrt{2}}\right)\right) - 0\right) + 2\pi$$

$$= 2 \times \left[8 - 8 \times \frac{\pi}{4}\right] + 2\pi$$

$$= 16 - 4\pi + 2\pi$$

$$= 16 - 2\pi \ u^2$$

(c)(i) $\int_0^e f(x)\,dx = \int_0^e \frac{x}{(x+e)^2}\,dx$

$u = x+e \Rightarrow x = u-e$

$\frac{du}{dx} = 1 \quad \therefore dx = du$

$x=0 \Rightarrow u=e$

$x=e \Rightarrow u=2e$

$= \int_{u=e}^{u=2e} \frac{u-e}{u^2}\,du$ ✓ as required.

(ii) $\int_0^e f(x)dx = \int_e^{2e} \frac{u-e}{u^2}\,du = \int_e^{2e}\left(\frac{u}{u^2} - \frac{e}{u^2}\right)du = \int_e^{2e}\left(\frac{1}{u} - eu^{-2}\right)du$.

$= \left[\ln u - e.\frac{u^{-1}}{-1}\right]_e^{2e} = \left[\ln u + \frac{e}{u}\right]_e^{2e}$

$= \left(\ln 2e + \frac{e}{2e}\right) - \left(\ln e + \frac{e}{e}\right) = \ln 2e + \frac{1}{2} - (1+1)$

$= \ln 2 + \ln e - \frac{3}{2}$

$= \ln 2 - \frac{1}{2}$

(d) (i) $\vec{OB} = \vec{OA} + \vec{OC} = \begin{pmatrix}-4+3\\3+4\end{pmatrix} = \begin{pmatrix}-1\\7\end{pmatrix}$

(ii) $\vec{OD} = \vec{OA} + \vec{AD} = \vec{OA} + \frac{2}{3}\vec{AB} = \vec{OA} + \frac{2}{3}\vec{OC}$.

$= \begin{pmatrix}-4\\3\end{pmatrix} + \frac{2}{3}\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}-4+\frac{2\times3}{3}\\3+\frac{2\times4}{3}\end{pmatrix} = \begin{pmatrix}-2\\3+\frac{8}{3}\end{pmatrix}$

$\vec{OD} = \begin{pmatrix}-2\\\frac{17}{3}\end{pmatrix}$

(iii) $\vec{OE} = (1-\lambda)\vec{OA} + \lambda\vec{OC} = (1-\lambda)\begin{pmatrix}-4\\3\end{pmatrix} + \lambda\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}(1-\lambda)\times-4+3\lambda\\(1-\lambda)\times3+4\lambda\end{pmatrix}$

$= \begin{pmatrix}-4+4\lambda+3\lambda\\3-3\lambda+4\lambda\end{pmatrix} = \begin{pmatrix}7\lambda-4\\\lambda+3\end{pmatrix}$ but E on $\vec{OB}$ ∴

∴ gradient OE = gradient OD

i.e $\frac{7\lambda-4}{\lambda+3} = \frac{-2}{\frac{17}{3}} = \frac{-6}{17} \Rightarrow 119\lambda - 68 = -6\lambda - 18$

$125\lambda = 50 \quad \therefore \lambda = \frac{50}{125} = \frac{2}{5}$

The End